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Reversibility of elementary cellular automata with fully asynchronous updating: An analysis of the rules with partial recurrence ☆

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ABSTRACT

Reversibility is a well-studied property for deterministic cellular automata but it is only partially explored when randomness appears in the update or in the transitions. Here, we present a detailed study of 21 finite-size elementary cellular automata with a fully asynchronous updating and periodic boundary conditions. We estimate their degree of reversibility by counting the number of recurrent configurations as a function of the size of the automaton (number of cells). We show that these rules exhibit various qualitative behaviours: some rules show a "strong reversibility" with a number of recurrent configurations that grows exponentially, while other rules have a "weak reversibility" with a linear growth. We also analyse the structure of the communication graph and report various scaling relations for the number of recurrent communication classes. We identify the cases where the divisibility of the size by 2 or 3 introduces some discontinuities in the behaviour. This study completes the characterisation of the 256 elementary cellular automata with regard to their recurrence properties.

1. Introduction

Cellular automata (CA) have gained their notoriety in the scientific landscape thanks to the simplicity of their structure: with a discrete space, time and set of states, the mathematical description of their evolution is simple and yet allows one to obtain a wide range of astonishing behaviours. Their mathematical structure assumes synchronous transitions, which, in some sense, amounts to the hypothesis that there is a global clock that synchronises the transitions of the cells that compose the automaton. Like any hypothesis in a model, the use of such a global clock is perfectly acceptable and sound, but it can also be enlightening to explore an asynchronous updating scheme, with cells applying their transitions randomly and independently. The last two decades have seen an increase in the interest in the study of asynchronous cellular automata with various questions raised about their dynamics and asymptotic behaviour [1–3]. The *fully asynchronous* updating, where a random cell is selected at each time step to update following a uniform distribution, and the *α-asynchronous* updating scheme, where each cell has a probability α to apply the rule at each time step, were the two most studied updating methods. Both have their pros and cons, which mainly depend on the context, but the fully asynchronous method has the clear advantage to simplify the analysis of the dynamics [1,4,5].

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In the CA literature, *reversibility* has been an important property to characterise the behaviour of a rule. It has been studied in the framework of dynamical systems and the first questions were raised in the 1970's [6–9] where the concept of *backward determinism* was proposed to capture the notion of reversibility: each configuration should have a unique predecessor, a property which amounts to the injectivity of the global transition function. However, this definition of reversibility cannot be transposed in a straightforward way to asynchronous cellular automata, as the non-determinism of the update implies that a configuration may have different successors and in general *does* have many successors.

Following this remark, in a Markovian setting where the state of the systems at time $t+1$ only depends on its state at time t, the notion of reversibility was identified with the property of *recurrence*, that is, the possibility to return almost surely an infinite number of times to the initial condition [10]. In other words, the notion of reversibility for asynchronous system captures an "eternal return" phenomenon. So far, only a few results have been obtained on this type of reversibility of asynchronous cellular automata [10–12,4,13]. Sarkar et al. studied the cases where the system can return back to the initial condition given a freely chosen updating sequence [11]. In a different direction, Wacker and Worsch have explored the possibility of constructing inverse rules for asynchronous automata systems [12]. Recently, Phung established the equivalence between reversibility, stable reversibility, and stable injectivity for asynchronous non-uniform cellular automata [13,14].

In this article, we consider the case of elementary cellular automata (ECA) under a fully asynchronous updating scheme. In a previous work, we have identified that 18 minimal representative ECAs, out of 88, are recurrent in the sense that they almost surely return to their initial condition an infinite number of times [10]. This study implied that the other rules are partially or totally irreversible, but the "degree of reversibility" of such systems was not evaluated. We thus now aim at quantifying this degree of reversibility by counting the number of recurrent configurations as a function of the size of the configuration. We are particularly interested in studying 21 minimal ECAs which do not show a reversible dynamics and where the classes of recurrence are not all reduced to a fixed point. In the next section we present the mathematical framework we use and then expose the main result in Section 3. The rules are then examined in order from those with a "strong reversibility" to those showing a "weak reversibility": in Section 4, we present the "almost recurrent" rules where the maximal number of recurrent configurations scales in 2^n. In Section 5 follows the analysis of the rules with a scaling in λ^n, where λ is non-trivial. Last, in Section 6 are exposed the rules with a weak reversibility, where the number of recurrent configurations is upper-bounded by a linear function. We then conclude with a few words of perspective (Sec. 7).

2. Elementary cellular automata

In this work, we consider one-dimensional binary cellular automata with nearest-neighbour interactions, a model commonly known as elementary cellular automata (ECA). We use periodic boundary conditions, which means that the cells are arranged as a ring; the set of cells' indices is denoted by $\mathcal{L} = \mathbb{Z}/n\mathbb{Z}$, where n is the number of cells. Here, we consider only the case $n > 3$ in order to avoid some exceptions. Each cell has a state in $\{0, 1\}$. The collection of all states at a given time is called a *configuration* and the set of configurations is denoted by $\mathcal{E}_n = \{0,1\}^{\mathcal{L}}$.

An ECA is defined by a local transition function $f : \{0,1\}^3 \to \{0,1\}$. This function defines how a cell updates its state according to its own state and the state of its left and right neighbours. A *transition* is a relation between a triplet (x, y, z) and the value $f(x, y, z)$. We call a transition *active* if it changes the state of a cell, *i.e.*, $f(x, y, z) \neq y$, otherwise, it is *passive*.

Here, the system is updated with a *fully asynchronous* method, that is, at each time step one cell is selected randomly and uniformly for update. The random sequence of cells that are selected for being updated is denoted by $(U_t)_{t\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers. The evolution of the fully asynchronous ECA from an initial condition x is represented by the stochastic process $(x^t)_{t\in\mathbb{N}}$ and defined recursively by: $x^0 = x$ and $x^{t+1} = F\ (x^t, U_t)$, with:

$$\forall i \in \mathcal{L}, x_i^{t+1} = \begin{cases} f(x_{i-1}^t, x_i^t, x_{i+1}^t) & \text{if } i = U_t, \\ x_i^t & \text{otherwise.} \end{cases}$$

In some previous works, transitions were called *rule min terms* (RMT) and an RMT (x, y, z) was indexed by its the number $r = 4x + 2y + z$. In this work, we use another code to identify the transitions: each transition is associated with a letter following the mapping shown on Table 1.

We will denote each ECA f with two complementary codes: a) its classical decimal code $\mathtt{W}(f) = f(0,0,0)\cdot 2^0 + f(0,0,1)\cdot 2^1 + \cdots + f(1,1,1)\cdot 2^7$. and, b) its *transition code* $\mathtt{T}(f)$, which consists of the concatenation of the letters in {A, ⋯, H} which denote the active transitions of f according to the code presented in Table 1. A rule will be identified by the couple $\mathtt{W}(f) - \mathtt{T}(f)$. To illustrate this notation, Table 1 shows the look-up table of the majority rule `232-DE`, in which the only active transitions are D and E, or, equivalently, RMT 2 and 5. This table also depicts the identity rule `204-I` (here, `I` stands for identity) where there is no active transition.

There are $2^8 = 256$ ECA rules, which can be reduced to 88 equivalence classes if we use the left-right symmetry and the exchange between 0's and 1's. The rules in each class which have the smallest number are called *minimal representative rules*.

The two homogeneous configurations all-0 and all-1 will be denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. For $n \in 2\mathbb{N}$, we define $\mathbf{01} = (01)^{n/2}$ and $\mathbf{10} = (10)^{n/2}$; and for $n \in 3\mathbb{N}$, we introduce $\mathbf{100} = (100)^{n/3}$, $\mathbf{010} = (010)^{n/3}$ and $\mathbf{001} = (001)^{n/3}$.

Definition 1. A configuration $x \in \mathcal{E}_n$ is called a fixed point if we have $F(x, u) = x$ for all $u \in \mathcal{L}$, that is, when all the cells of x are passive. We denote by $\mathcal{FP}_n$ the set of fixed points of an ECA of size n.

Table 1
Table of transitions and their associated labels with two different presentations (RMT order and transition code order). Examples: look-up tables of ECA `232-DE` and ECA `204-I` (identity rule).

(a) classical ordering (RMT)

(x,y,z)	000	001	010	011	100	101	110	111	
RMT	0	1	2	3	4	5	6	7	
t-code	A	B	E	F	C	D	G	H	rule
f(x,y,z)	0	0	0	1	0	1	1	1	`232-DE`
f(x,y,z)	0	0	1	1	0	0	1	1	`204-I`

(b) transition code ordering (A,..., H)

(x,y,z)	000	001	100	101	010	011	110	111	
t-code	A	B	C	D	E	F	G	H	rule
RMT	0	1	4	5	2	3	6	7	
f(x,y,z)	0	0	0	1	0	1	1	1	`232-DE`
f(x,y,z)	0	0	0	0	1	1	1	1	`204-I`

3. Convergence, recurrence and main result

3.1. General definitions

The evolution of our system can be described by a Markov chain over $\mathcal{E}_n$. We now define the recurrence properties of our asynchronous cellular automata using the classical tools from Markov chain theory. We aim at separating the *recurrent* configurations from the *transient* ones. Intuitively, a *transient* configuration is such that there exists a particular sequence of updates which brings the system to a configuration from which it will never be possible to return to the initial configuration; by contrast, if the system starts from a *recurrent* configuration, it will return infinitely often to this configuration.

Given two configurations $x, y \in \mathcal{E}_n$, we say that y is a *successor* of x if there exists an update $u \in \mathcal{L}$ such that $F(x,u) = y$. By iterating the successor relation (zero times or more), we obtain the relation *reachable*: for two arbitrary configurations $x, y \in \mathcal{E}_n$, we say that y is reachable from x if y can be obtained from x by following a sequence of successor relations, that is, for $k \geq 1$, $\exists k \in \mathbb{N}, \exists (x^i)_{i\in\{0,\dots,k\}}$ such that $x^0 = x$, $x^k = y$ and $\forall i \in \{0,\cdots,k-1\}, \exists u_i \in \mathcal{L}; x^{i+1} = F(x^i, u_i)$. By definition every configuration is reachable from itself. A configuration that is not reachable from any other configuration than itself is an *isolated fixed point*. A configuration that is not the successor of any configuration (including itself) is *inaccessible*.

We can now define the recurrence property:

Definition 2. A configuration $x \in \mathcal{E}_n$ is recurrent if for every configuration y that is reachable from x, x is also reachable from y. A non-recurrent configuration is transient.

Definition 3. A rule is recurrent for size n if each configuration of $\mathcal{E}_n$ is recurrent. We say that a rule is recurrent if it is recurrent for all the sizes $n > 3$.

More generally, if y is reachable from x and x is reachable from y, we say that x and y *communicate*. Clearly, the relationship "communicate" is an equivalence relation; this relation partitions the set of configurations into *communication classes*.

Consequently, two major behaviours exist: for transient configurations, the system remains a finite time in its communication class, then "escapes" this class and never returns back to it. By contrast, when the system starts from a recurrent configuration, it remains in its the communication class of this configuration forever; we call such a class a *recurrent communication class*, or *recurrent class* for short.

Our aim is now to determine, given a cellular automaton f and a ring size n, the number of *recurrent configurations* $\mathcal{R}(n)$. In the case where the growth of $\mathcal{R}$ is exponential, we will be particularly interested in finding the *growth factor* of rule, that is, the constant γ such that $\mathcal{R}(n) \sim \gamma^n$. In a second step, we will also evaluate the scaling relations that govern $\mathcal{C}(n)$, the number of recurrent classes.

3.2. Recurrent rules

In our previous work [15,10], we have identified the ECAs which are recurrent (see Theorem 1 below). Here, we have $\mathcal{R}(n) = 2^n$. Moreover, we also identified the number of communication classes $\mathcal{C}(n)$ for these recurrent rules: these results are summarised below.

Theorem 1 *([15,10]). Among the 88 minimal ECA rules, the following 18 rules are recurrent:*

33-ADEFGH	*35-ABDEFGH*	*38-BDFGH*	*41-ADEGH*	*43-ABDEGH*
46-BDGH	*51-ABCDEFGH*	*54-BCDFGH*	*57-ACDEGH*	*60-CDGH*
62-BCDGH	*105-ADEH*	*108-DH*	*134-BFG*	*142-BG*
150-BCFG	*156-CG*	*204-I*		

Moreover, the scaling relationship of the number of the communication classes, which are all recurrent classes, are given by:

ECAs	$\mathcal{C}(n)$
33, 35, 41, 43, 51, 57	1
38, 46, 54, 60, 62	2
134, 142, 150	linear : $\begin{cases} 3+n/2 & \text{if } n \in 2\mathbb{Z}, \\ 2+\lfloor n/2 \rfloor & \text{otherwise.} \end{cases}$
105	linear : $\begin{cases} 2 & \text{if } n \in 2\mathbb{Z}+1, \\ n/2+3 & \text{if } n \in 4\mathbb{Z}, \\ n/2 & \text{otherwise.} \end{cases}$
108, 156, 204	exponential

3.3. The special case of the convergent rules

As a special case where it is relatively easy to evaluate the recurrence properties, there are rules where the fixed points are the only recurrent configurations and where the other configurations are transient.

Definition 4. A rule is convergent for size n if its recurrent configurations are all fixed points and if all the other configurations are transient. It is convergent if it is convergent for all the sizes $n > 3$.

The convergent rules were identified in a previous work [5,4,16] and we found that there are 50 minimal convergent ECA rules (see Theorem 2 below). We also proposed a recursive procedure, called the *fixed point graph*, for identifying and counting the number of fixed points of the ECAs [5,17]. For the convergent rules, all the fixed points form an isolated recurrent communication class and we thus have $\mathcal{C}(n) = |\mathcal{FP}_n|$.

Theorem 2 *([5,4]). Among the 88 minimal ECA rules, the following 50 rules are convergent:*

0-EFGH	2-BEFGH	4-FGH	5-AFGH	8-EGH
10-BEGH	12-GH	13-AGH	18-BCEFGH	24-CEGH
26-BCEGH	32-DEFGH	34-BDEFGH	36-DFGH	40-DEGH
42-BDEGH	44-DGH	50-BCDEFGH	56-CDEGH	58-BCDEGH
72-EH	74-BEH	76-H	77-AH	78-BH
90-BCEH	94-BCH	104-DEH	106-BDEH	122-BCDEH
128-EFG	130-BEFG	132-FG	136-EG	138-BEG
140-G	146-BCEFG	152-CEG	154-BCEG	160-DEFG
162-BDEFG	164-DFG	168-DEG	170-BDEG	172-DG
178-BCDEFG	184-CDEG	200-E	204-I	232-DE

This result allows us to decide the case of many rules: indeed, all these rules but the Identity (ECA `204-I`) are non-recurrent. Obviously, we thus have $\mathcal{R}(n) = \mathcal{C}(n) = |\mathcal{FP}_n|$.

Now, our subject of interest is the remaining 21 minimal ECAs. The following theorem depicts the main result.

Theorem 3 *(Main result). There are 21 non-recurrent non-convergent minimal ECAs. The number of their recurrent configurations $\mathcal{R}(n)$ and their number of recurrent classes $\mathcal{C}(n)$ verify the relations expressed in Table 2.*

We identified seven types of behaviour (A1 to C2); the analysis of these classes is exposed in the three following sections.

4. Quasi-reversible rules

In this section, we present the rules with a growth factor of 2, or for which there exists an exponential growth of $\mathcal{R}(n)$ in 2^n when we exclude some particular cases (here, $n \in 3\mathbb{N}$). Figs. 1 and 3 show typical examples of space-time diagrams produced by these rules, which are divided into two sets, A1 and A2. One can observe that for type A1 rules, the system has a highly noisy evolution, with no structure appearing, which easily explains that almost all configurations will be visited. For type A2 rules, one can see that the evolution resembles some noisy arborescent structures which develop on a background formed of a repetition of patterns `001`. Let us now analyse these rules formally.

4.1. Type A1

Theorem 4. *For the following ECAs `3-ABEFGH, 11-ABEGH, 19-ABCEFGH, 25-ACEGH, 27-ABCEGH` the number of recurrent configurations is $\mathcal{R}(n) = 2^n - 1$ and the number of recurrent classes is $\mathcal{C}(n) = 1$.*

Table 2
Synthetic presentation the different sets of rules with the number of their recurrent configurations and recurrent classes.

Type	ECA	$\mathcal{R}(n)$	$\mathcal{C}(n)$
A1	3-ABEFGH 11-ABEGH 19-ABCEFGH 25-ACEGH 27-ABCEGH	2^n - 1	1
A2	37-ADFGH 45-ADGH	$2^n - 1$ ($n \notin 3\mathbb{N}$) or 3 ($n \in 3\mathbb{N}$)	1 ($n \notin 3\mathbb{N}$) or 3 ($n \in 3\mathbb{N}$)
B1	1-AEFGH 9-AEGH 129-AEFG 137-AEG	$\sim \phi^n$, where ϕ is the golden ratio	1 or 2
B2	28-CGH 29-ACGH	$\sim \lambda^n$, with $\lambda \approx 1.52$	$\sim \mu^n$, with $\mu \approx 1.32$
B3	73-AEH	$\sim \lambda^n$, with $\lambda \approx 1.84$	$\sim \mu^n$, with $\mu \approx 1.47$
C1	6-BFGH 14-BGH 22-BCFGH 30-BCGH	2 (even size) or $2n$ (odd size)	2 (even size) or 1 (odd size)
C2	7-ABFGH 15-ABGH 23-ABCFGH	2 (even size) or $2n$ (odd size)	3 (even size) or 2 (odd size)

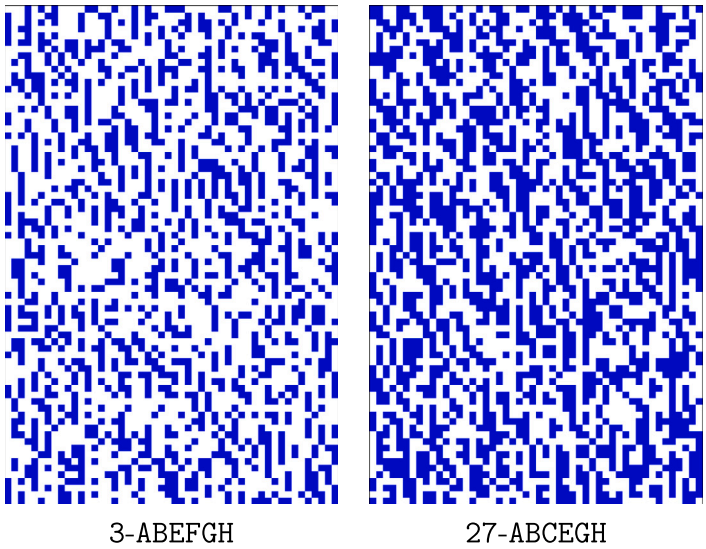


Fig. 1. Space-time diagrams of ECA 3-ABEFGH and 27-ABCEGH (type A1) for a size $n = 50$. White and blue squares represent cells in states 0 and 1, respectively. Time goes from bottom to top and each line represents the state of the automaton after n random updates. This convention is kept in the rest of the text.

Remark that A, E, G, and H are active for all the rules considered. Moreover, D is passive and B or C is also active for all these rules. The theorem relies on the following properties.

Proposition 1. *The homogeneous configuration* $\mathbf{1}$ *is an inaccessible configuration, that is, it is not the successor of any configuration, including itself.*

Proof. This is clear since the application of transition D is the only way to reach $\mathbf{1}$, but this transition is passive. Moreover, note that since transition H is active, $\mathbf{1}$ is not its own successor. □

Proposition 2. *The configuration* $\mathbf{0}$ *is reachable from any configuration.*

Proof. Observe that for any configuration x different from $\mathbf{0}$, it is possible to find an update that will make a 1 disappear. Indeed, for $x = \mathbf{1}$, transition H applies in all cells. For a configuration $x \neq \mathbf{1}$, observe that if there is a pattern 11 present in the configuration, then transition G is applicable. If x contains no pattern 11 it means that the 1's are isolated and transition E can be applied to decrease the number of 1's. □

To structure the following proofs, we introduce the notion of a 1-region (resp. 0-region): it corresponds to a maximal set of contiguous cells with state 1 (resp. 0) in a given configuration.

Proposition 3. *All the configurations different from* $\mathbf{1}$ *are reachable from* $\mathbf{0}$.

Proof. Let us prove that we can find an update pattern that transforms $\mathbf{0}$ in a given configuration x. Let $i_1, i_2, \cdots$ be the positions of the cells which are the left frontiers of a 1-region in x. In other words, for each i_k, we have $x_{i_k-1} = 0$ and $x_{i_k} = 1$.

It can be noted that, by construction, the values i_k are separated by at least two (modulo n). We can thus update the cells with index i_k independently to transform them into 1's. Note that transition A always applies because we will always find a 0 surrounded by two 0's. We thus obtain a configuration x' that has as many 1-regions as x, and each of this 1-region is an isolated 1.

Recall that for all the rules considered, either transition B or C is active; without loss of generality, let us assume that transition C is active. In a second step, we can note that for each region, using transition C we can make the 1-region of x' grow by updating the cells sequentially from left to right in order to make the regions of the current configuration coincide with the regions of x. This is done by using transition C only, until we reach x when the last region has been made equal to the last region of x.

In the case where transition B is active, the proof is similar by the left-right symmetry. □

As a consequence of Proposition 1, 2, and 3, for all the rules considered in the theorem, we have only a single recurrent communication class ($\mathcal{C}(n) = 1$), which consists of all the configurations but $\mathbf{1}$, which ends the proof of Theorem 4.

4.2. Type A2

Theorem 5. *For the following ECAs* 37-ADFGH, 45-ADGH *the number of recurrent configurations is:* $\mathcal{R}(n) = \begin{cases} 3 & \text{for } n \in 3\mathbb{N}, \\ 2^n - 1 & \text{for } n \notin 3\mathbb{N}, \end{cases}$

and the number of recurrent classes is: $\mathcal{C}(n) = \begin{cases} 3 & \text{for } n \in 3\mathbb{N}, \\ 1 & \text{for } n \notin 3\mathbb{N}. \end{cases}$

This result relies on the following properties, which are all stated for these two rules, where the transitions A, D, G and H are active.

Proposition 4. *For* $n \in 3\mathbb{N}$, $\mathbf{010}$ *is a fixed point which is reachable from any configuration.*

Proof. As transition B, C, and E are passive, the cells of $\mathbf{010}$ are passive and this is a fixed point.

For $n \in 3\mathbb{N}$, let us consider an arbitrary configuration α and show that $\mathbf{010}$ is reachable from α. First, let us show that α can reach a configuration β which contains the pattern 010. We make the following disjunction:

a) if α contains 01, then simply remark that β either contains 010, and we are done, or it contains 011, in which case we update the third cell in order to make a 010 appear. Indeed, since transitions G and H are both active, the central cell of $11q$ is active for both $q = 0$ and $q = 1$.

b) if α does not contain 01, then it is either the homogeneous configuration $\mathbf{0}$, in which case any update creates a pattern 010, or it is the homogeneous configuration $\mathbf{1}$, in which case two updates separated by two cells, that is, cell i and $i+2$ for an arbitrary $i \in \mathcal{L}$, makes the pattern 010 appear.

In a second step, let us consider the pattern $010xyz$ in β and let us demonstrate that the pattern xyz can be turned into 010. There are eight cases for the value of xyz. Let us denote these cases by (a), (b), ... (h) according to whether transitions A, B, ..., H respectively apply in xyz.

(a) $xyz = 000$: As transition A is active, we simply update the central cell.

(b) $xyz = 001$: This case is more delicate since the central cell is passive. We propose to apply the following sequence of updates:
$010\ 001 \xrightarrow{A} 010\ 101 \xrightarrow{D} 011\ 101 \xrightarrow{D} 011\ 111 \xrightarrow{G} 011\ 110 \xrightarrow{G} 011\ 100 \xrightarrow{G} 011\ 000 \xrightarrow{G} 010\ 000 \xrightarrow{A} 010\ 010.$

(c) $xyz = 100$: Here too, several updates are needed:
$010\ 100 \xrightarrow{D} 011\ 100 \xrightarrow{G} 011\ 000 \xrightarrow{G} 010\ 000 \xrightarrow{A} 010\ 010.$

(d) $xyz = 101$: As transition D is active, updating the central cell sends us to case (h).

(e) $xyz = 010$: The goal is already attained.

(f) $xyz = 011$: As mentioned above, remark that the third cell (z) is necessarily active since transitions G and H are active. Updating this cell makes a 010 appear.

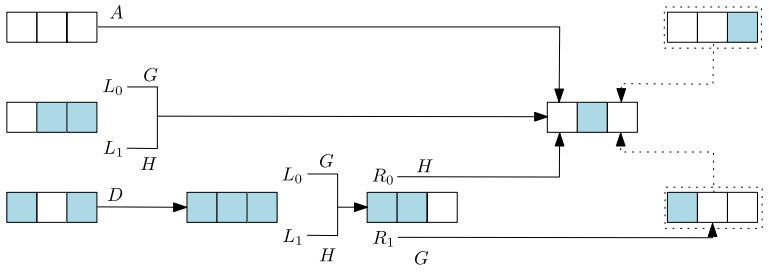


Fig. 2. Transition from different blocks of states xyz to block 010 with active transition A, D, G, and H. White and blue box represent cells in states 0 and 1 respectively.

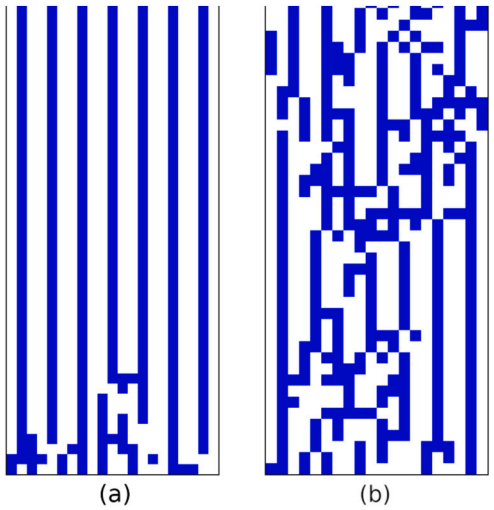


Fig. 3. Space-time diagrams of ECA 37-ADFGH (type A2) (a) left: $n = 21$, in this example the system reaches a fixed point; (b) right: $n = 20$ in this case, all the configurations except **0** are recurrent.

(g) $xyz = 110$: As transition G is active, updating the central cell sends us back to case (c).
(h) $xyz = 111$: As seen above, the third cell is necessarily active; updating this cell sends us back to case (g).

The various links between cases and the associated transformations are depicted in Fig. 2. By iterating this operation, each block of three cells can thus be turned into an 010, which shows that **010** is reachable from β, and thus also reachable from α. □

We have thus shown that for $n \in 3\mathbb{N}$, the only recurrent configurations are the three fixed points **001**, **010** and **100**.
Let us now study the general case when $n \notin 3\mathbb{N}$.

Proposition 5. *The homogeneous configuration* **0** *is inaccessible.*

This is clear since the application of E is the only way to reach **0**, but this transition is passive. Moreover, since transition A is active, **0** is not its own successor.

Proposition 6. *For $n \notin 3\mathbb{N}$, the configuration* **1** *is reachable from any configuration.*

Proof. This property can be established by observing that for any configuration x different from **1**, it is possible to find an update pattern that will increase the number of 1's.

The increase in the number of 1's is immediate if x contains a pattern 000 or 101, that is, if transitions A or D can be applied.

Let us assume that this is not the case, that is, x contains no 0 surrounded by two 0's (000) and no isolated 0 (101). Note that x necessarily contains a pattern 00; otherwise all the 0's of x would be isolated, which would contradict the previous hypothesis.

Let us assume that x contains the pattern 11. Since $x \neq \mathbf{1}$, it also necessarily contains the pattern 110, and also the pattern 1100, as the last 0 cannot be isolated (101). In this case, remark that we can apply transition G on the second cell, and then transition A on the third cell, and then transition D on the second cell, which gives a pattern 1110 and adds a 1 to the configuration.

The last case to consider is when x contains none of the patterns 000, 101, or 11. Remark that each 00 is necessarily followed by a 1, and that each 001 is necessarily followed by another 001 pattern (because the last 1 can only be followed by 00). This leads

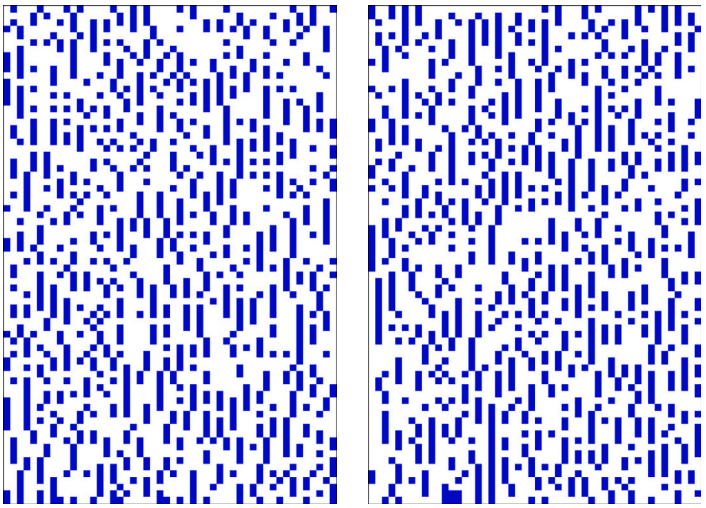


Fig. 4. Space-time diagrams of ECA 9-AEGH and 137-AEG (type B1).

to a contradiction with $n \notin 3\mathbb{N}$. Indeed, if x is of the form $x = (100)^k\alpha$, then $\alpha = 0$ and $\alpha = 1$ are both impossible because x would contain 000 or 11; and if $x = (100)^k 0\beta$, then, again, the two possibilities for β lead to a contradiction. This ends the proof of the proposition. □

Proposition 7. *For $n \notin 3\mathbb{N}$, any configuration different from* **0** *is reachable from* **1**.

This property is the symmetric of Proposition 3 by the conjugation operation, *i.e.*, exchanging 0's and 1's. Here, note that transitions G and H are active.

We can now notice that all the configurations but **0** communicate with configuration **1**. This implies that all the configurations of $\mathcal{E}_n \backslash \{\mathbf{0}\}$ are recurrent. We thus have $\mathcal{R}(n) = 2^n - 1$ for $n \notin 3\mathbb{N}$.

Following the discussion above, $\mathcal{C}(n) = 1$ when $n \notin 3\mathbb{N}$. On the other hand, $\mathcal{C}(n) = |\mathcal{FP}| = 3$ when $n \in 3\mathbb{N}$. The ring size thus introduces a huge discrepancy, with two qualitatively different behaviours of the system.

5. Rules with a non-trivial exponential number of recurrent configurations

In this section, we present the rules which display an exponential number of recurrent configurations, but with an irrational non-trivial growth factor. Figs. 4 and 6 display examples of typical space-time diagrams produced by such rules: for type B1, the evolution is noisy and shows no stable part (pairs 11 are avoided). For types B2 and B3, it can be seen that the dynamics of such rules results from the combination of two effects: the creation of stable zones in the configuration (stripes in the space-time diagram) and a noisy evolution in the interval between such zones (Fig. 6).

5.1. Type B1

Theorem 6. *For the following ECAs: 1-AEFGH, 9-AEGH, 110-BDH, and 126-BCDH, the number of recurrent configurations is $\mathcal{R}(n) \sim \phi^n$, where ϕ is the golden ratio and the number of recurrent classes is $\mathcal{C}(n) = 1$ or $\mathcal{C}(n) = 2$.*

Let us denote by ϱ the set of (circular) configurations which do not contain the pattern 11.

Proposition 8. *For the rules 1-AEFGH and 9-AEGH, respectively, 129-AEFG and 137-AEG, the set of recurrent configurations is ϱ, and, respectively, $\varrho \cup \{\mathbf{1}\}$.*

Proof. In order to prove the proposition above, we proceed in three steps.

(a) Let us show that for 1-AEFGH and 9-AEGH, **0** is reachable from any configuration and for 129-AEFG and 137-AEG, **0** is reachable from any configuration different from **1**.

To see why, simply observe that from any configuration $x \notin \{\mathbf{0}, \mathbf{1}\}$, it is possible to find an update that will make a 1 disappear. Indeed, observe that there exists a cell $i \in \mathcal{L}$, such that $x_i = 1$ and $x_{i+1} = 0$. Such a cell i is active, as either transition E or transition G will be active on it. For the rules 1-AEFGH and 9-AEGH, **0** is also reachable from **1**, as this configuration is unstable, while for rules 129-AEFG and 137-AEG, it is a fixed point, and thus a recurrent configuration, as transition H is passive.

(b) In a second step, remark that any configuration of ϱ can be attained from $\mathbf{0}$. Indeed, we can sequentially update the cells in which we want the 1's to appear: transition A is always active in such cells as they are necessarily a 0 surrounded by two 0's.

(c) A configuration of $\varepsilon_n \setminus \varrho$ cannot be attained from $\mathbf{0}$. This third property results from the fact that a pair 11 cannot appear by updating a configuration that does not contain it. Indeed, there are only three possibilities to make a 11 appear: 101 → 111, 001 → 011, or 100 → 110, but the three transitions that would respectively apply, namely D, B, and C, are passive. □

To establish Theorem 6, we simply need to count the size of ϱ. Writing $s_n = \operatorname{card} \varrho(n)$, we have: $s_{n+1} = s_n + s_{n-1}$ with $s_2 = 3$ and $s_3 = 4$. This series is called the Lucas numbers and it is the sequence A000032 in the On-Line Encyclopedia of Integer Sequences (OEIS). We have: $s_n = \phi^n + (-\phi)^{-n}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Note that the result directly applies to the minimal representative rules 110-BDH, and 126-BCDH, as they are equivalent to the rules 129-AEFG and 137-AEG by conjugation.

Also note that we have only one recurrent class ($\mathcal{C}(n) = 1$) for 1-AEFGH and 9-AEGH, and two recurrent classes for 110-BDH and 126-BCDH ($\mathcal{C}(n) = 2$). The communication graph of 9-AEGH is shown on Fig. 5 for a size $n = 6$.

5.2. Type B2

Theorem 7. *For the following ECAs 28-CGH, 29-ACGH the number of recurrent configurations verifies $\mathcal{R}(n) \sim \lambda^n$, where $\lambda \approx 1.52$ and the number of recurrent classes verifies $\mathcal{C}(n) \sim \mu^n$, where $\mu \approx 1.32$.*

This result relies on the following property.

Proposition 9. *For the rules 28-CGH, 29-ACGH, a configuration which contains the pattern 111, 1100, or 1000 is transient.*

Proof. Remark that C, G, and H are active for the two rules considered.

(a) Let us assume that a configuration x contains a 111. One can see easily that since transition H is active, its application will increase the number of 1-regions, however, this number can never decrease as the transitions D and E are passive. It will thus not be possible to reach x again.

(b) If a configuration contains a 1100, then by applying transition C we create a pattern 111, which amounts to the previous case.

(c) Similarly, if a configuration contains a 1000, then by applying the transition C twice, we reach a configuration with a 111. □

Proposition 10. *For the rules 28-CGH, 29-ACGH, the configurations which do not contain any of the patterns 111, 000, or 1100 are recurrent.*

Proof. It can be observed that if a configuration x does not contain one of the three patterns 111, 000, or 1100, then it can be decomposed into a sequence of parts where each part is itself a sequence of length-2 patterns 10 followed by a sequence of 100 or 110 length-3 patterns. In other words, we can write:

$$x = (01)^{i_1}(0q_1 1)^{j_1} \dots (01)^{i_k}(0q_k 1)^{j_k},$$

where $i_1, j_1, \dots, i_k, j_k$ are arbitrary positive integers, k is a non-negative integer, and where $q_1, \dots q_k$ are arbitrary (binary) states.

In such a configuration x, one can see that the only active cells are those where the transition C ($q_i = 0$) or transition G ($q_i = 1$) apply. However, this transformation only changes the state q_i, keeping the updated cell active and not creating or removing any other neighbouring active cell. As a consequence, if a configuration y can be reached from x then x can also be reached from y, which implies that x is recurrent. The communication class of a configuration $x = (01)^{i_1}(0q_1 1)^{j_1} \dots (01)^{i_k}(0q_k 1)^{j_k}$ is thus formed by the set of configurations of size 2^k:

$$C_x = \left\{(01)^{i_1}(0q_1 1)^{j_1} \dots (01)^{i_k}(0q_k 1)^{j_k}; (q_1, \dots q_k) \in \{0,1\}^k\right\}. \quad \square$$

Remark, that the homogeneous configuration $\mathbf{0}$ is a special case:

- For 28-CGH, transition A is passive, therefore, $\mathbf{0}$ is a fixed point, and as transition E is passive, this is an isolated fixed point, which implies that it is a recurrent configuration.
- For 29-ACGH, $\mathbf{0}$ is transient because transition A is active, which implies that a pattern 1000 can be created; in other words, it can reach a transient configuration and is therefore transient itself (see above Proposition 9).

To count the recurrent configurations, remark that a configuration is recurrent if it avoids the pattern 111, 000, and 1100. Let s_n be the number of such configurations of size n; we have $s_n = s_{n-2} + 2s_{n-3}$ with $s_1 = 0$, $s_2 = 2$, $s_3 = 6$ [18]. The first elements of the sequence are $0, 2, 6, 2, 10, 14, 14, 34, 42$. The sequence has index A072328 in the On-Line Encyclopedia of Integer Sequences (OEIS) and we have $s_n \sim \lambda^n$, where $\lambda \approx 1.52$ is the real root of $P(x) = x^3 - x - 2$.

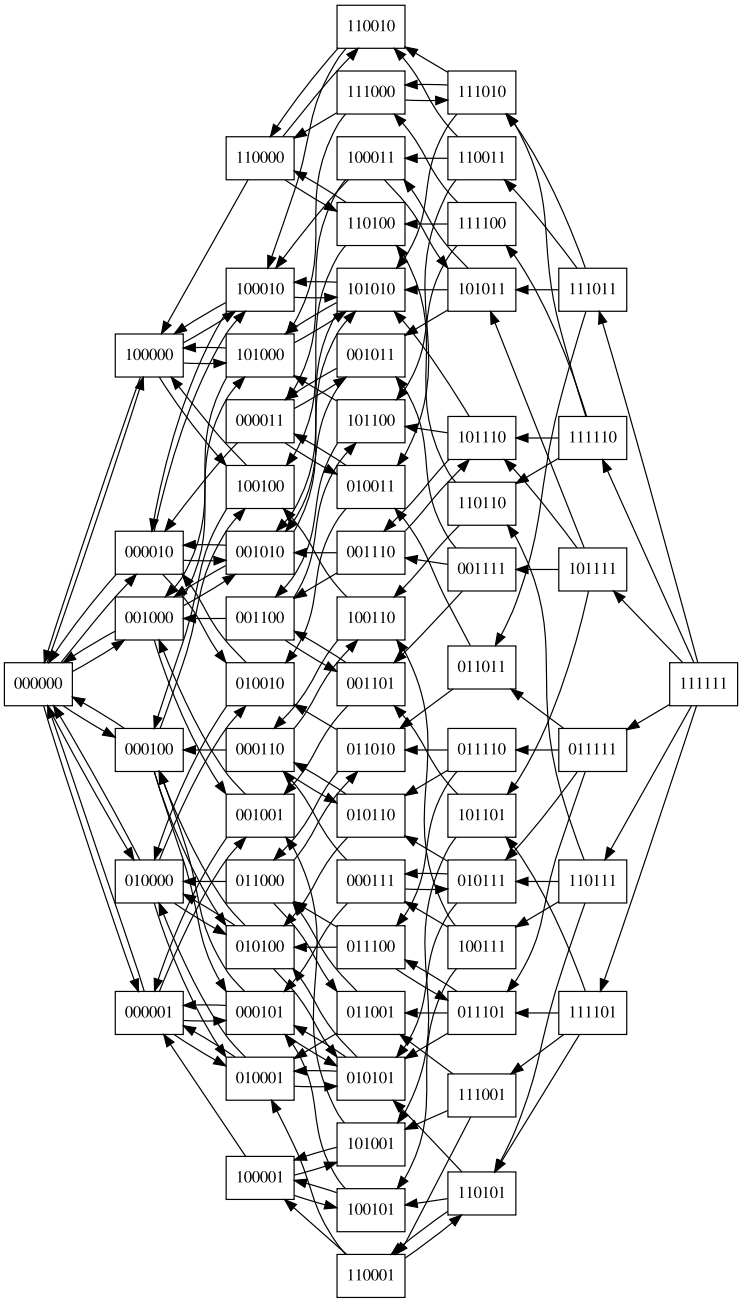


Fig. 5. Communication graph for ECA 9-AEGH. An arrow between two configurations indicates a potential transition of the system. Self-loops are removed for the sake of clarity. Note that recurrent states are the configurations which do not contain two consecutive 1's.

On the other hand, the number of recurrent classes equals the number of partitions of n into blocks of length 2 (pattern 10) and length 3 (patterns 100 and 110). The number of recurrent classes $\mathcal{C}(n)$ thus follows the recurrence equation $\mathcal{C}(n) = \mathcal{C}(n-2) + \mathcal{C}(n-3)$ and it has the index A182097 in the OEIS. We have $\mathcal{C}(n) \sim \mu^n$ with $\mu \approx 1.32$, which is the real root of $P(x) = x^3 - x - 1$.

5.3. Type B3

Theorem 8. *For the ECA 73-AEH, the number of recurrent configurations verifies $\mathcal{R}(n) \sim \lambda^n$, where $\lambda \approx 1.84$ and the number of recurrent classes verifies $\mathcal{C}(n) \sim \mu^n$, where $\mu \approx 1.47$.*

This result relies on the following property.

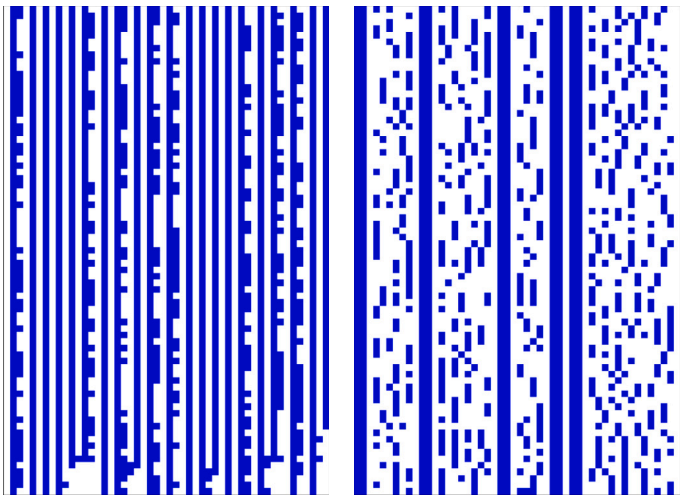


Fig. 6. Space-time diagrams of ECA 28-CGH (type B2) and ECA 73-AEH (type B3).

Proposition 11. *For the rule* 73-AEH*, a configuration which contains the pattern* 111 *is transient.*

Proof. Let us assume that a configuration x contains a 111. One can see easily that since transition H is active, from x, one can reach a configuration y which does not contain any 111. Then, remark that from y, we can only increase the number of 1's by using active transition A. However, this creates isolated 1's and can never produce a 111 pattern (since B, C and D are passive). Therefore, it will not be possible to reach x again. □

Proposition 12. *For the rule* 73-AEH*, a configuration which does not contain the pattern* 111 *is recurrent.*

Proof. It can be observed that if a configuration x does not contain the pattern 111, then either it is the all-zero configuration $\mathbf{0}$ or all the 1's it contains are isolated or grouped by two. Formally, up to a given number of shifts, we can write:

$$x = (0110)w_1(0110)w_2 \dots (0110)w_k,$$

where $w_1, \dots, w_k$ are words in which all the 1's are isolated.

In such a configuration, the patterns with 0110 are static, as transitions B, C, D, F and G are passive. Inside the words w_i, only A and E apply, and these two transitions imply *direct reversibility*: if x can reach y by updating a given cell, then y can reach x by updating the same cell. So if x can reach y' with a given sequence of updates, y' can reach x by applying this sequence in reverse order, which implies that x is recurrent. □

To establish Theorem 8, we simply need to count s_n, the number of configurations of size n which do not contain the pattern 111. By applying classical combinatorial techniques, we have: $s_{n+1} = s_n + s_{n-1} + s_{n-2}$ with $s_1 = 1$, $s_2 = 3$, $s_3 = 7$ [18]. This sequence has the index A001644 in the OEIS; its first elements are $1, 3, 7, 11, 21, 39, 71$ and we have $s_n \sim \lambda^n$, where $\lambda \approx 1.84$ is the real root of the polynomial $P(x) = x^3 - x^2 - x - 1$.

To count the number of recurrent classes, simply remark that every configuration x that has the form $x = (0110)w_1(0110)w_2 \dots (0110)w_k$ described above communicates with a configuration $r(x) = (0110)0^{|w_1|}(0110)0^{|w_2|} \dots (0110)0^{|w_k|}$, where $|w|$ denotes the length of a word w. The number of recurrent classes $\mathcal{C}(n)$ is thus formed by the number of circular words which avoid the patterns 010 and 111. Indeed, such configurations correspond to all the possible forms for $r(x)$. The same enumeration techniques lead to the following recurrence equation $\mathcal{C}(n) = \mathcal{C}(n-2) + \mathcal{C}(n-3)$, which implies that $\mathcal{C}(n) \sim \mu^n$, where $\mu \approx 1.47$ is the unique real root of $P(x) = x^3 - x^2 - 1$. This corresponds to the A001609 sequence in the OEIS and its first elements are $1, 1, 4, 5, 6, 10, 15, 21, 31$.

6. Weakly reversible rules

In this section, we present the rules where the reversibility is weak (type C1 and C2). For such rules, the maximal number of recurrent configurations is a linear function. Fig. 7 presents a typical example of such evolution: for rule 15-ABGH, one can see that the system is attracted towards one of the two fixed points **01** or **10**, and that in the case of an odd size, this fixed point is not reachable from other configurations; this creates a remaining default which travels right and periodically loops.

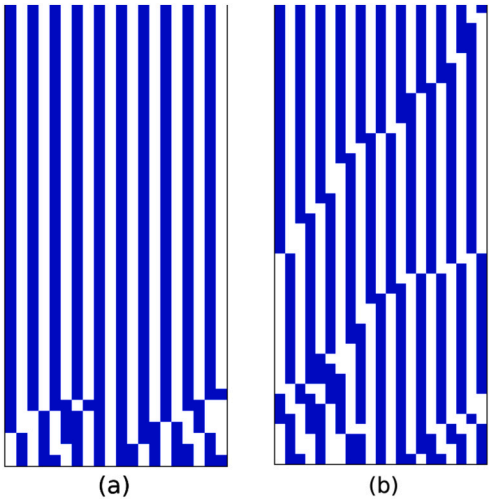


Fig. 7. Space-time diagrams of ECA 15-ABGH (type C2): (a) $n = 20$, the system evolves to a fixed point; (b) $n = 21$ the system converges to a set of recurrent configurations (see text).

Theorem 9. *For the following ECAs 6-BFGH, 7-ABFGH, 14-BGH, 15-ABGH, 22-BCFGH, 23-ABCFGH, 30-BCGH,*

the number of recurrent configurations is: $\mathcal{R}(n) = \begin{cases} 2 & \text{for } n \in 2\mathbb{N}, \\ 2n & \text{for } n \in 2\mathbb{N}+1, \end{cases}$

and the number of recurrent classes is: $\mathcal{C}(n) = \begin{cases} 2 \text{ or } 3 & \text{for } n \in 2\mathbb{N}, \\ 1 \text{ or } 2 & \text{for } n \in 2\mathbb{N}+1. \end{cases}$

We remark that for $n \in 2\mathbb{N}$, **01** and **10** are fixed points, since transitions D and E are passive. We also note they are the configurations with a maximal number of 1-regions.

Let us denote by η the set of configurations which are either equivalent to $001(01)^{\frac{n-3}{2}}$ or to $011(01)^{\frac{n-3}{2}}$, up to shifts. We remark that for an odd size ($n \in 2\mathbb{N}+1$), η is the set of configurations with a maximal number of 1-regions.

Theorem 9 relies on the following property.

Proposition 13. *If $x \neq \mathbf{0}$ is a configuration with k 1-regions, where k is not maximal, then there exists a configuration y with $(k+1)$ 1-regions which is reachable from x.*

Proof. Let us examine a configuration $x \neq \mathbf{0}$ where the number of 1-regions k is not maximal, that is, with have $k < \lfloor \frac{n}{2} \rfloor$. (An example of such a configuration is 0011010101.)

Remark that B, G and H are active for all the rules considered.

If x contains the pattern 111, then we can directly apply H to increase the number of 1-regions.

If x contains the pattern 0001, observe that we can apply the transition B twice to create a pattern 111, and we are sent to the previous case.

In the case where x does neither contain 0001 nor 111, then it necessarily contains the pattern 0011 (otherwise it would have a maximal number of 1-regions). We can thus apply the active transition G to obtain a configuration which contains the pattern 0001, which sends us to the previous case. □

Moreover, if transition A is passive, then **0** is an *isolated fixed point*, as transition E is passive. It is thus a recurrent configuration. If A is active, then **0** is not recurrent: it communicates with the pattern 0001, and this case has been discussed above.

Let us now examine the case with an odd size.

Proposition 14. *For $n \in 2\mathbb{N}+1$, let η be the set of configurations with $k = \lfloor \frac{n}{2} \rfloor$ 1-regions. Any configuration of η is reachable from any configuration of η.*

Proof. Let x_i be a configuration of η which is equivalent by shifts to $001(01)^{\frac{n-3}{2}}$ and where i indicates the position of 001. Similarly, we define y_i as the configuration which is equivalent by shifts to $011(01)^{\frac{n-3}{2}}$ and where i indicates the position of 011.

Let us denote by $x \rightarrowtail y$, the fact that y is reachable from x.

Let us take a configuration $x_i \in \eta$ and show that: (a) $x_i \rightarrowtail y_i$; (b) $x_i \rightarrowtail x_{i+2}$. (Recall that the indices of the cells are taken modulo n.)

Table 3
Sequence of updates showing the communication between the configurations of η. See the text for the definitions of x_i and y_i.

(x_1)	001010101 cbededede	→	(y_1)	011010101 dfgdedede	→	(x_3)	010010101 decbededе	→	(y_3)	010110101 dedfgdede	→
(x_5)	010100101 dedecbede	→	(y_5)	010101101 dededfgde	→	(x_7)	010101001 dededecbe	→	(y_7)	010101011 dedededfg	→
(x_9)	010101010 bedededec	→	(y_9)	110101010 fgdededed	→	(x_2)	100101010 ecbededed	→	(y_2)	101101010 edfgdeded	→
(x_4)	101001010 edecbeded	→	(y_4)	101011010 ededfgded	→	(x_6)	101010010 edededcbed	→	(y_6)	101010110 edededfgd	→
(x_8)	101010100 edededecb	→	(y_8)	101010101 gdedededf	→	(x_1)	001010101 cbededede				

We advise the readers to follow the text below with an eye on Table 3. Also note that according to the hypotheses of Theorem 9, transitions B and G are active for all the rules considered.

(a) The property $x_i \rightarrowtail y_i$ is obtained with transition B, which allows us to transform a 001 into a 011.

(b) The property $y_i \rightarrowtail x_{i+2}$ is obtained with transition G, which allows us to transform a 011 into a 001. Observe that the pattern 001 has moved two cells to the right.

As the cells are disposed on a ring, all the configurations of η communicate.

Also note that since transitions D and E are passive, the number of 1-regions cannot decrease.

For $n \in 2\mathbb{N}+1$, the recurrent configurations are the members of the set η, which are $2n$. When n is even, the only recurrent configurations are the two fixed points **01** and **10**. □

According to the discussion above, for ECAs 7-ABFGH, 15-ABGH, and 23-ABCFGH, as transition A is active, we have only one recurrent class ($\mathcal{C}(n)=1$) when n is odd and two recurrent classes ($\mathcal{C}(n)=|\mathcal{FP}|=2$) when n is even. Moreover, when A is passive, as a special case, the configuration **0** creates an additional isolated recurrent class. Therefore, for the ECAs 6-BFGH, 14-BGH, 22-BCFGH, 30-BCGH, we have $\mathcal{C}(n)=2$ when n is odd and $\mathcal{C}(n)=3$ when n is even.

7. Conclusion

To sum up, with this last series of results, the situation of the 256 ECA is now fully covered with regard to their main properties of recurrence. We extended the techniques presented for recurrent systems [5,4,15,10] to the case where the recurrence was only partial. We fully characterised the 21 remaining non-recurrent non-convergent ECA rules, counting their number of recurrent configurations and number of recurrent classes.

Taking the scaling laws of the number of recurrent configurations as a criterion, we distinguished three main types of behaviour linked to the partial recurrence: (a) almost recurrent rules, with a maximal number of recurrent configurations that scales in 2^n; (b) partially recurrent rules, with a scaling in λ^n where λ is non-trivial, and (c) weakly reversible rules, where the number recurrent configurations is upper-bounded by a linear function.

Among the different cases, some remarkable behaviours emerged; for instance, rules of type A2 and type C showed a convergent dynamics when the ring size is divisible by three and two, respectively. We also noted that four rules (type B1) showed a scaling law with the golden ratio as a growth factor, while other type-B rules displayed a growth factor which is the solution of a polynomial of order 3.

Moreover, we studied the structure of the communication graph and estimated the number of recurrent classes, which also allowed us to exhibit various scaling relations. We found that some rules had a constant number of recurrent classes while other rules, such as ECA 28-CGH, 29-ACGH, and 73-AEH this number grows exponentially.

Natural extensions of this work on the effects of asynchronism and the reversibility property raise the following questions:

- What can be said for larger class of rules? For instance, for radius-2 rules, ternary systems, two-dimensional rules, etc.
- In a previous work, we explored the structure of the communication classes of the recurrent rules and examined if all the configurations of the communication class were “equally exposed” in the sense that they would have equal frequencies of appearance in the recurrent set [19]. Can we conduct a similar analysis of the “exposition” characteristics for the recurrent classes of the 21 non-convergent non-recurrent rules?
- What can be said about the extension of these results to other asynchronous updating schemes, for instance α-, β- or γ-asynchronism [20], or delay-sensitive [21] updating schemes?

CRediT authorship contribution statement

Souvik Roy: Writing – original draft, Methodology, Formal analysis, Conceptualization. **Nazim Fatès:** Writing – original draft, Formal analysis, Conceptualization. **Sukanta Das:** Writing – review & editing, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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