Ahmedabad University

School of Engineering and Applied Sciences

MAT 248 Applied Linear Algebra

Mid Term Examination 2023-24 (Monsoon Semester)

Total Marks = 60 Time – 2 hours

Enrolment Number: .....

No. of printed pages: 4

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Print your name: Signature:

Instructions:

- Each question is compulsory.
- Calculator is not allowed.
- Handwritten notes and the text-book are allowed.
- Use of any electronic device is strictly forbidden.

1. (a) In the vector space $C[0,\pi]$, let f, g, h and k be the vectors defined by

$$f(x) = 1, g(x) = x, h(x) = \cos x, k(x) = \cos^2 \frac{x}{2}$$

for $0 \leq x \leq \pi$. Prove or disprove: $\{f, g, h, k\}$ is a linearly dependent set (Justify). (Marks: 4)

(b) If A is an $n \times n$ matrix, then sum of the n-eigen values of A is the trace of A and the product of n-eigen values of A is the determinant of A. (Marks: 4+4=8)

(c) Let U, V be finite dimenssional vector spaces over $\mathbb{R}$. Consider a linear transformation $T : U \to V$. Prove that if $\dim U > \dim V$, then T cannot be injective (one to one). (Marks: 3)

2. Show that the determinant of following 4×4 matrix is non-zero. (Marks: 15)

$$\begin{bmatrix} 2023^{2023} & 2024^{2024} & 2022^{2022} & 2020^{2020} \\ 2020^{2020} & 2021^{2021} & 2022^{2022} & 2024^{2024} \\ 2018^{2018} & 2020^{2020} & 2019^{2019} & 2026^{2026} \\ 2020^{2020} & 2018^{2018} & 2022^{2022} & 2017^{2017} \end{bmatrix}$$

In order to show the determinant is non-zero, you may use the following ideas as guide.

Like vector space, a **FIELD** is a non-empty set F with two operations addition(+) and multiplication (.) such that for any $x, y \in F$, there exist unique elements $x + y$ and $x.y$ in F that are subjected to the following conditions.

(a) $x + y = y + x$

(b) $(x + y) + z = x + (y + z)$

(c) $\exists\, 0$ such that $x + 0 = x$

(d) $\forall x, \exists(-x)$ such that $x + (-x) = 0$

(e) $xy = yx$

(f) $(x + y).z = xz + yz$ and $x(y + z) = xy + yz$

(g) $(x.y).z = x.(y.z)$

(h) $\exists 1 \neq 0$ such that $1.x = x$

(i) $\forall x \neq 0, \exists x^{-1}$ such that $x.x^{-1} = 1$.

Known examples of **FIELD** consist of real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$. Note that the set $F = \{odd, even\}$ also constitutes an example of a **FIELD** where we can define addition and multiplication as follows:

$$(odd + odd) = even,\ (odd + even) = odd$$
$$(even + even) = even,\ (odd.odd) = odd$$
$$(odd.even) = even,\ (even.even) = even$$

The **SCALERS** in linear algebra can be chosen from any **FIELD**. The familiar concepts in linear algebra such as matrix, linear system, determinant, vector space, linear independence can be defined over any **FIELD**.

3. (a) Now you know the definition of **FIELD**, show that $\mathbb{Q}$ with usual addition and multiplication form a field.
(Marks: 2)

(b) Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional. (Marks: 13)

Hint: There are many ways to attack this problem. One way is countability argument. Note that finite cartesian product of countable set is countable.

4. $V = M_{2\times 2}(\mathbb{R})$ denote the vector space of real valued matrices.

(a) Show that the map

$$T : V \to V$$

given by $T(A) = A - A^T$ is a linear transformation. (Marks: 5)

(b) Compute its characteristic equation. (Marks: 2)

(c) Compute its Eigenvalues and the corresponding Eigenvectors. (Marks: 8)