Ahmedabad University

# School of Arts and Sciences

## MAT442 Complex Analysis

**Mid Term Examination 2021-22 (Winter Semester)**

**Total Marks = 100** **Time - 2 hours**

Name & Enrolment Number:
Date:
No. of printed pages: 3

1. (10 points) Find the image of the triangle, with vertices located at $z_1 = 0$, $z_2 = 1 + i$ and $z_3 = 1 - i$, under the mapping $f(z) = z^2$. Draw a rough sketch of the image.

2. (10 points)

   a. The principal value of the complex logarithm function is defined as $Log(z) = \log_e(|z|) + iArg(z)$, where $Arg(z)$ is the principal value of the argument of $z$, with $-\pi < Arg(z) \leq \pi$. Show that this function is not continuous at $z = -k$ where $k > 0$ is any positive real number.

   b. Explain the concept of "branch cut" for $Log(z)$ function.
   (Hint: Consider restricting the domain of $Log(z)$ to make this function continuous.)

3. (10 points)

   a. Let $z = x + iy$ such that $e^z = e$. Find all the possible values of $x$ and $y$.

   b. Is it true that $Log(i^9) = 9\,Log(i)$? Here $Log$ is the principal branch of complex logarithm (as defined in the previous problem).

4. (10 points) Let
$$f(x + iy) = \frac{xy}{x^2 + y^2}, \quad \text{for } x \neq 0, y \neq 0.$$

Compute the limit $\lim_{x+iy\to 0+i0} f(x+iy)$ if it exists. Otherwise, show that the limit does not exist.

5. (10 points) What is the period of each of the following complex functions?

   a. $e^{\pi+z}$.

   b. $e^{\pi z}$.

6. (10 points)

   a. Find the principal value of $\sqrt{(1+i)}$

   b. State the Gauss-Lucas theorem and illustrate it by giving an example.

7. (10 points) Prove that the power series $\sum_{n=0}^{\infty} na_n z^{n-1}$ has the same radius of convergence as the power series $\sum_{n=0}^{\infty} a_n z^n$.

8. (10 points) Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(3z)^n}{n}.$$

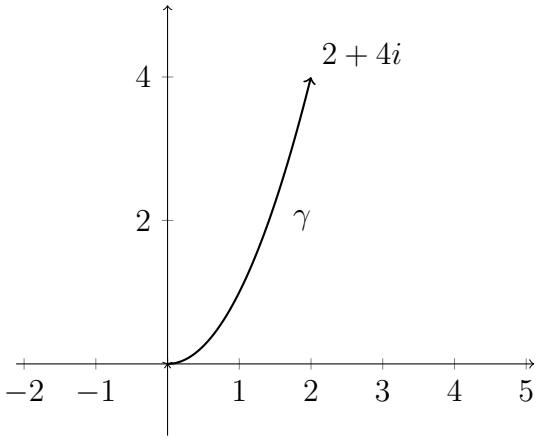
9. (10 points) Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(4n!)}{(2n!)(n!)^2} z^n.$$

10. (10 points) Evaluate the integral

$$\int_{\gamma} \bar{z}\, dz,$$

where $\gamma$ is the portion of the parabola $y = x^2$ from $x = 0$ to $x = 2$ (see the Fig. below).




11. (10 points) Bonus Problem:

a. Let $\mathbb{M}$ be the set of all $2 \times 2$ real matrices of the special form as given below. Let

$$\mathbb{M} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

It is known that $\mathbb{M}$ is a field with addition and multiplication defined as matrix addition and matrix multiplication, respectively. Prove that the field of complex number $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$ is isomorphic to $\mathbb{M}$. What is the image of the complex number $i$ under this isomorphism.
(Hint: Two fields $F$ and $K$ are called isomorphic, if there exists a bijective map

$$f : F \longrightarrow K,$$

such that

$$f(x + y) = f(x) + f(y), \quad \text{and } f(x \cdot y) = f(x)f(y), \quad \text{for all } x, y \in F.)$$

b. Let $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$. Using the $\epsilon$ -$\delta$ method show that

$$\lim_{z \to z_0} f(z) + g(z) = A + B.$$

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**End of Exam**.

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