Ahmedabad University

School of Arts and Sciences

MAT465/565 Fourier Analysis And Its Applications

Mid Term Examination 2022-23 (Monsoon Semester)

Total Marks = 100 **Time - 2 hours**

Name & Enrolment Number:
Date:
No. of printed pages: 3

1. (15 points) Let $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ be the Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi < x \leq \frac{-\pi}{2} \\ 1 & \frac{-\pi}{2} < x \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \leq \pi \end{cases}$$

   (a) Find the Fourier coefficients a_n and b_n for all n.

   (b) i. Find $S(\frac{-3\pi}{4})$.
   ii. Find $S(\frac{-\pi}{2})$.
   iii. Find $S(\frac{\pi}{4})$.
   iv. Find $S(\frac{\pi}{2})$.
   v. Find $S(\frac{3\pi}{4})$.

2. (15 points) Answer the following questions.

   (a) Compute the Fourier series by periodically extending the function $f(x) = x^2$ defined on $(-\pi, \pi]$.

(b) State Parseval's identity for Fourier Series, and use it to compute

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

3. (10 points) In this problem the goal is to prove that finite Fourier series of degree N gives the best (trigonometric) approximation of that order in $L^2([0,1])$ to a function. For this assume $f(t)$ to be in $L^2([0,1])$. If α_1, α_2, ..., α_n are any given complex numbers, then show that

$$\left\| f - \sum_{n=-N}^{N} \langle f, e_n \rangle e_n \right\| \leq \left\| f - \sum_{n=-N}^{N} \alpha_n e_n \right\|,$$

where $e_n = e^{i2\pi nt}$.

4. (10 points) Determine the complex numbers c_0, c_1, c_2 and c_3 in order to make the following integral as small as possible.

$$\int_0^{\pi} |c_0 + c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x - \cos^4 x|^2 \, dx.$$

5. (15 points) Use Parseval's identity for a periodic extension of the function

$$f(x) = \frac{\pi e^{-iax}}{\sin(\pi a)}, \quad -\pi < x \leq \pi,$$

to compute

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}.$$

Here a is a given real number that is not an integer.

6. (15 points) Find the Fourier transforms of the following functions

a

$$f(x) = e^{-|x|}.$$

b

$$\Pi(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

7. (20 points) It is known that the Fourier transform of $f(x) = e^{-\pi x^2}$ is given by $F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} \, dx = e^{-\pi s^2}$. Find the Fourier transforms of the following functions

(a) $f(x)\cos(2\pi rt)$,

(b) $f(x-r)$,

(c) $f(rx)$,

(d) $rf'(x)$

where r is a non-zero real number.

8. (10 points) (**Bonus Problem**) Suppose that f has period 1 and $\int_0^1 |f(x)|\,dx < \infty$.

   (a) If a is a real number show that

$$\int_a^{a+1} f(x)\,dx = \int_0^1 f(x)\,dx$$

   (b) Assume $g(x) = f(x+a)$ for all x. Assume that the Fourier series for $f(x)$ is known and is given as $\sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}$. Show that the Fourier series of $g(x)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n\, e^{i2\pi na}\, e^{i2\pi nx}.$$

End of Exam.
