MAT268 MATHEMATICAL BIOLOGY Midsem Examination*

Total marks = 20; Time = 3 Hours

September 2023

Instructions: You are being evaluated on the presentation and the correctness of your answers. Try to answer questions in a clear, direct, and efficient way. Sloppy or incorrect use of technical terms will lower your mark. There are three questions (1-3), each of 10 marks. You can choose to answer any two. If you choose to attempt all three, then the best two of the three would be considered for you.

1. To evaluate the stability of a fixed point in a discrete model of population dynamics,

$$P_{t+1} = F(P_t),$$

we looked at $\frac{\delta_{t+1}}{\delta_t}$, where δ_t denotes small perturbation from a fixed point P^* at time t, such that $P_t = P^* + \delta_t$. We found that this ratio is equal to $F'(P^*)$, i.e. the first derivative at the fixed point P^*.

   a. Using the Taylor expansion,

$$F(P_t) = F(P^*) + \delta_t F'(P^*) + \frac{\delta_t^2}{2} F''(P^*) + \frac{\delta_t^3}{6} F'''(P^*) + ...,$$

   find the general expression for δ_{t+1} in terms of δ_t and the higher order derivatives up to the third order.

   b. Let $\alpha = F'(P^*)$. Use the above formula to calculate δ_t up to $t = 5$ for $\alpha = 0.5$, starting from $\delta_0 = 0.1$. Write down your calculations in form of a table for $t = 1$ to 5. Comment on the stability of P^* based on your calculations. Note: For $|\alpha| < 1$, you may ignore the higher order derivatives in the formula you derived in a, but comment why.

   c. Repeat this for $\alpha = -0.5$. What special difference do you observe in terms of how perturbations grow or decay.

   d. For $\alpha = 0$, repeat the same using the second order derivative term $\beta = F''(P^*)$ calculated at the fixed point for the model. Let $\beta = 0.2$. Comment on the stability of the fixed point in this situation. Note: Ignore the third order derivative and above.

   e. Repeat the process for $\alpha = 1$ and $\beta = -0.2$. You can ignore values after the third decimal if it becomes too tedious.

   f. Amongst all the cases that you worked out, in which of the cases the perturbations decay with time? In which case the change is the fastest and where it is slowest?

2. You are given the following three models.

$$\Delta N = N(1 - N)(aN - 1) \tag{1}$$

$$\Delta N = -N(1 - N)(1 - \frac{N}{a}) \tag{2}$$

$$\Delta N = N(1 - N)(a - N) \tag{3}$$

For all the three models,

*You can use your class notes. Photocopied notes from others are not allowed. Use of mobile calculator is permitted.

a. Plot the graphs of ΔN versus N.

b. Find the fixed points using the graphs as well as analytically.

c. Determine stability of all the fixed points using graphical method.

d. Find the range of a for which your conclusions are valid for all the three models.

e. Based on your analysis, which one of the model does represent an Allee effect in the population?

3. The following model describes a certain coyote population. Two stage classes – pup and adult – are used while the matrix

$$P = \begin{bmatrix} 0.3 & 2 \\ 0.2 & 0 \end{bmatrix} \tag{4}$$

describes changes over a time step of 1 year.

a. Explain what each entry in this matrix is saying about the population.

b. Calculate the eigen values and eigen vectors of the matrix.

c. What do the eigen values say about the dynamics of the population?

d. Express the initial vector $X_0 = (1, 1)$ as a sum of the eigen vectors.

e. Using this expression, compute X_t for t = 1 to 3. Based on your observations, if you were to simulate this model using computer, draw how would your plot of pups and adult population with time look like.